WHAT EVERY YOUNG MATHLETE SHOULD KNOW

(from *www.moems.org*, the official web-site for Math Olympiads for Elementary and Middle Schools)

1 Vocabulary and Language

The following explains, defines, or lists some of the words that may be used by authors of the Olympiad problems. The definitions below are the official definitions for the Olympiad contests.

1.1 Basic Terms

Sum, **difference**, **product**, **quotient**, **ratio** of (two) numbers.
Square of a number.
Factors of a number.
The **value** of a number is the simplest name for that number.
Division M **Square root** of a number, **cube** of a number.

1.2 Reading

Read "1 + 2 + 3 + ..." as "one plus two plus three and so forth".
Read "1 + 2 + 3 + ... + 10" "as one plus two plus three and so forth *up to ten*."

1.3 Standard Form of a Number

The **standard form** of a number refers to the form in which we usually write numbers (also called Hindu-Arabic numerals or positional notation).

A **digit** is any one of the ten numerals 0,1,2,3,4,5,6,7,8,9. Combinations of digits assigned place values are used to write all numbers. A number may be described by the number of digits it contains: 358 is a three-digit number. The "**lead-digit**" (leftmost digit) of a number is not counted as a digit if it is 0: 0358 is a *three-digit* number. **Terminal zeros** of a number are the zeros to the right of the last nonzero digit: 30500 has two terminal zeros because to the right of the digit 5 there are two zeros.

1.4 Sets of numbers

Whole Numbers = $\{0, 1, 2, 3, ...\}$
Natural or Counting Numbers = $\{1, 2, 3, ...\}$
Division M **Integers** = $\{..., 3, 2, 1, 0, +1, +2, +3, ...\}$
Division M **Negative numbers** are all numbers less than zero.
Consecutive Numbers are natural numbers that differ by 1, such as 83, 84, 85, 86, and 87.
Consecutive Even Numbers are multiples of 2 that differ by 2, such as 36, 38, 40, and 42.
Consecutive Odd Numbers are nonmultiples of 2 that differ by 2, such as 57, 59, 61, and 63.

1.5 Divisibility

Let a and b be natural numbers. Then a is **divisible** by b if b divides a with zero remainder (or $\frac{a}{b}$ is equal to a natural number). In such instances b is called a **factor** of a, and a is called a **multiple** of b.

1.6 Number Theory

a. A **prime number** is a natural number that has exactly two different factors, namely itself and 1. Note that 1 is not a prime number. *Examples*: 2, 3, 5, 7, 11, 13, ...

b. A **composite number** is a natural number which has more than two different factors, namely 1, itself, and at least one other factor. *Examples*: 4, 6, 8, 9, 10, 12, ... (Thus, there are 3 categories of natural numbers: prime, composite, and 1.)

c. A number is **factored completely** when it is expressed as a product of prime numbers. *Example*: $144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3$. It may also be written as $144 = 2^4 \times 3^2$.

d. The **Greatest Common Factor (GCF)** of two natural numbers is the largest natural number that divides each of the two given numbers with zero remainder. *Example*: $GCF(12, 18) = 6$.

e. If the GCF of two numbers is 1, then we say the numbers are **relatively prime** or **co-prime**.

f. The **Least Common Multiple (LCM)** of two natural numbers is the smallest number that each of the given numbers divides with zero remainder. *Example*: $LCM(12, 18) = 36$.

g. **Order of Operations**. When computing the value of expressions involving two or more operations, the following priorities must be observed from left to right:

1. Do operations in parentheses, braces, or brackets,
2. Do multiplication and division from left to right, and then
3. Do addition and subtraction from left to right.

Example:
$$\begin{array}{rl} & 3 + 4 \times 5 - 8 \div (9-7) \\ = & 3 + 4 \times 5 - 8 \div 2 \\ = & 3 + 20 - 4 \\ = & 19 \end{array}$$

1.7 Fractions

a. A **common (or simple) fraction** is a fraction in the form $\frac{a}{b}$ where a and b are whole numbers and b cannot be 0.

b. A **unit fraction** is a common fraction with numerator 1.

c. A **proper fraction** is a common fraction in which $a < b$. Its value is between 0 and 1.

d. An **improper fraction** is a common fraction in which $a \geq b$. Its value is 1 or greater than 1. A fraction whose denominator is 1 will be accepted in place of an integer.

e. A **complex fraction** is a fraction whose numerator or denominator contains a fraction. *Examples*: $\frac{\frac{1}{3}}{5}$, $\frac{7}{\frac{3}{8}}$, $\frac{3-\frac{1}{3}}{3+\frac{1}{3}}$

f. The fraction is **simplified** ("**in lowest terms**") if a and b have no common factor other than 1 ($GCF(a, b) = 1$).

g. <u>*Division M*</u> A **decimal** or **decimal fraction** is a fraction whose denominator is a power of ten. The decimal is written using decimal point notation. *Examples*: $.7 = \frac{7}{10}$; .36; .005; 1.4

h. <u>*Division M*</u> A **percent** or **percent fraction** is a fraction whose denominator is 100, which is represented by the percent sign. *Examples*: $\frac{45}{100} = 45\%; 8\%; 125\%; 0.3\%$

1.8 Statistics and Probability

The **average** (**arithmetic mean**) of a set of N numbers is the sum of all N numbers divided by N. The **mode** of a set of numbers is the number listed most often. The *median* of an ordered set of numbers is the middle number if N is odd, or the mean of the two middle numbers if N is even.

The **probability** of an event is a value between 0 and 1 inclusive that expresses how likely an event is to occur. It is often found by dividing the number of times an event *does* occur by the total number of times the event *can* possibly occur. *Example*: The probability of rolling an odd number on a die is $\frac{3}{6}$ or $\frac{1}{2}$. Either $\frac{3}{6}$ or $\frac{1}{2}$ will be accepted as a correct probability.

1.9 Geometry

a. Angle: degree-measure.

b. Kinds of angles: acute, right, obtuse, straight, reflex.

c. Polygons:

Triangles: acute, right, obtuse, scalene, isosceles, equilateral. *Note: an equilateral triangle is isosceles with all sides equal.*

Quadrilaterals: parallelogram, rectangle, square, trapezoid, rhombus. *Note: a square is a rectangle with all sides equal. It is also a rhombus with all angles equal in measure.*

Others: pentagon, hexagon, octagon, decagon, dodecagon, icosagon.

Area: the number of unit squares contained in the interior of a region.

Perimeter: the number of unit lengths in the boundary of a plane figure.

Circumference: the perimeter of a circular region.

Congruent figures: two or more plane figures whose corresponding sides and angles have the same measure.

Similar figures: two or more plane figures whose size may be different but whose shape is the same. *Note: all squares are similar; all circles are similar.*

Division M:

Geometric Solids: Rectangular Solid, Cube, Right Circular Cylinder.

Volume: the number of unit cubes contained in the interior of a solid.

Surface Area: the sum of the areas of all the faces of a geometric solid.

2 Skills

2.1 Computation

The tools of arithmetic are needed for problem solving. Competency in the basic operations on whole numbers, fractions, and decimals is essential for success in problem solving at all levels. In *Division M* competency in basic operations on integers and signed numbers should be developed.

2.2 Answers

Unless otherwise specified in a problem, equivalent numbers or expressions should be accepted. For example, $3\frac{1}{2}$, $\frac{7}{2}$, and 3.5 are equivalent.

Units of measure generally are not required in answers but must be correct if given in an answer. Measures of area are usually written as square units, sq. units, or units^2. For example, square centimeters may be abbreviated as *sq.cm*, or $cm \times cm$, or cm^2. In *Division M*, cubic measures are treated in a like manner.

After reading a problem, a wise procedure is to indicate the nature of the answer at the bottom of a worksheet before starting the work necessary for solution.

Examples: "$A =$ ____, $B =$ ____"; "The largest number is ____".

Another worthwhile device in practice sessions is to require the student to write the answer in a simple declarative sentence using the wording of the question itself. *Example: "The average speed is 54 miles per hour."* This device usually causes the student to reread the problem.

2.3 Measurement

The student should be familiar with units of measurement for time, length, area, and weight (and for *Division M*, volume) in English and metric systems. Within a system of measurement, the student should be able to convert from one unit to another.

3 Some Useful Theorems

1. If a number is divisible by 2^n, then the number formed by the last n digits of the given number is also divisible by 2^n; and conversely.
Example: $7,292,536$ is divisible by 2 (or 2^1) because 6 is divisible by 2.
Example: $7,292,536$ is divisible by 4 (or 2^2) because 36 is divisible by 4.
Example: $7,292,536$ is divisible by 8 (or 2^3) because 536 is divisible by 8.

2. If the sum of the digits of a number is divisible by 9, then the number is divisible by 9.
If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.
Example: $658,773$ is divisible by 9 because $6+5+8+7+7+3=36$ which is a multiple of 9.
Example: $323,745$ is divisible by 3 because $3+2+3+7+4+5=24$ which is a multiple of 3.

3. A number is divisible by 5 if its units digit is 5 or 0.

4. A number is divisible by 11 if the difference between the sum of the odd-place digits and the sum of the even-place digits is 0 or a multiple of 11.
Example: $90,728$ is divisible by 11 because $(9+7+8)-(0+2)=24-2=22$, which is a multiple of 11.

5. If A and B are natural numbers, then:

(i) $GCF(A,B) \times LCM(A,B) = A \times B$.
(ii) $LCM(A,B) = (A \times B) \div GCF(A,B)$.
(iii) $GCF(A,B) = (A \times B) \div LCM(A,B)$.

Example: If $A=9$ and $B=12$: $GCF(9,12)=3$, $LCM(9,12)=36$, $A \times B = 9 \times 12 = 108$. Then: (i) $3 \times 36 = 108$; (ii) $108 \div 3 = 36$; (iii) $108 \div 36 = 3$.

6. If p represents a prime number, then p^n has $n+1$ factors.

Example: $2 \times 2 \times 2 \times 2 \times 2 = 2^5$ has 6 factors which are 1, 2, 2×2, $2 \times 2 \times 2$, $2 \times 2 \times 2 \times 2$, $2 \times 2 \times 2 \times 2 \times 2$.
In **exponential form**, the factors are: 1, 2, 2^2, 2^3, 2^4, and 2^5.
In **standard form**, the factors are: 1, 2, 4, 8, 16, and 32.
Notice that the factors of 2^5 include 1 and 2^5 itself.

Problem: how many factors does 72 have?

$72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$. Since 2^3 has 4 factors and 3^2 has 3 factors, 72 has $4 \times 3 = 12$ factors. The factors may be obtained by multiplying any one of the factors of 2^3 by any one of the factors of 3^2: $(1, 2, 2^2, 2^3) \times (1, 3, 3^2)$. Written in order, the 12 factors are: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

4 Some General Strategies for Problem Solving

1. Draw a picture or diagram
2. Make an organized list
3. Solve a simpler problem
4. Work backward
5. Find a pattern
6. Make a table
7. Guess, check and revise
8. Use reasoning (logic)

Try to guess, check, and revise when no other method presents itself. With time and practice, more efficient strategies should start to present themselves.

Thorough discussions of these and many other useful topics may be found in *Creative Problem Solving in School Mathematics 2nd Edition* and *Math Olympiad Contest Problems for Elementary and Middle Schools*.